$$G_0(z) = G_1(z) + 2a = a + \frac{\lambda az}{\sqrt{1 - i\lambda z}\, X(z)} \frac{1}{\pi i} \int_0^\infty \frac{X^-(t)}{\sqrt{1 + i\lambda t}} \frac{dt}{t - z} \tag{2.18}$$

vanishes at infinity. Its boundary values $G_0^+(t)$ will represent the Fourier transform of the desired function $v'(t)$

$$v'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_0^+(\tau)\, e^{-it\tau}\, d\tau \tag{2.19}$$

The preceding expression determines according to Eqs. (1.14) a piecewise holomorphic function $\varphi(z)$ which satisfies all requirements of the problem.

The author is very grateful to G. A. Dzhanashiia and R. D. Bantsuri for useful discussions of the content of this section.

The problem of internal semi-infinite stringer in an unbounded plate is solved in an analogous manner [3, 4].

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ON THE DEVELOPMENT OF CAVITIES IN VISCOUS BODIES

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The problem of the development of cavities in viscous bodies under infinite deformations is considered. A formation of the problem of the development of a cavity under the conditions of a stationary slow flow of a viscous Newtonian fluid is given in Sect. 1. An exact solution of the problem of broadening of the cavity from the initial viscous one is obtained in Sect. 2. The analysis is limited to the case of the plane problem.

1. Viscous body. Let us consider a viscous body subjected to Newton's law and occupying an infinite domain in the exterior of some contour L (the problem is considered a plane one). The interior of the contour L is some cavity whose shape is known only at the initial instant of load application. The following are assumed: (a) The cavity walls are subjected to a constant pressure $p(t)$; (b) there is a homogeneous state of stress $\sigma_x = \sigma_x^\infty(t)$, $\sigma_y = \sigma_y^\infty(t)$, $\tau_{xy} = 0$ (t is time) at infinity; (c) the contour of the cavity at any time has two axes of symmetry which coincide with the axes of the fixed Cartesian coordinate system; (d) the flow is slow and quasi-stationary so that inertial terms can be discarded in the Navier-Stokes equations. For simplicity we limit ourselves to the case of an incompressible body; however, this assumption is not essential to the subsequent exposition.

The stress tensor components σ_x, σ_y, τ_{xy} and the velocity vector components u, v in the xy coordinate system can be represented in the case under consideration by using formulas analogous to the Koslov-Muskhelishvili relations in the plane elasticity theory problem

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z, t) \qquad (z = x + iy)$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z, t) + \Psi(z, t)] \tag{1.1}$$
$$2\mu(u + iv) = \varphi(z, t) - z\overline{\varphi'(z, t)} - \overline{\psi(z, t)} \qquad (\Phi = \varphi', \ \Psi = \psi')$$

Here 2μ is the shear viscosity coefficient, $\varphi(z, t)$ and $\psi(z, t)$ are single-valued analytic functions of z in the domain occupied by the body; the prime above the letter denotes the derivative with respect to the appropriate comples variable.

The kinematic compatibility condition

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0 \tag{1.2}$$

($F(x, y, 0)$ is a given function)

should be satisfied on the unknown boundary of the cavity whose equation has the form $F(x, y, t) = 0$.

Moreover, the condition

$$\varphi(z, t) + z\overline{\varphi'(z, t)} + \overline{\psi(z, t)} = -p(t)z \qquad (z \in L) \tag{1.3}$$

should be satisfied on the contour L.

At the infinitely distant point the functions $\varphi(z, t)$ and $\psi(z, t)$ behave thus as $z \to \infty$:

$$\varphi(z, t) = {}^1/_4[\sigma_x^\infty(t) + \sigma_y^\infty(t)]z + O(z^{-1}) \tag{1.4}$$
$$\psi(z, t) = {}^1/_2[\sigma_y^\infty(t) - \sigma_x^\infty(t)]z + O(z^{-1})$$

Therefore, the formulated problem reduces to the boundary value problem (1.2)–(1.4).

Let us go over to the exterior of a unit circle in the parametric ζ plane by using the mapping $z = \omega(\zeta, t)$; the analytic function $\omega(\zeta, t)$ maps the domain $|\zeta| > 1$ conformally on the exterior of the contour L with a mutually one-to-one correspondence between the infinitely distant points, as well as the corresponding positions of the real and imaginary axes. Therefore

$$\omega(\zeta, t) = c(t)\zeta + O(\zeta^{-1}) \qquad \text{as} \quad \zeta \to \infty \tag{1.5}$$

where $c(t)$ is a real function.

The boundary conditions (1.3), (1.4) on the ζ plane are hence written as follows:

$$\varphi_*(\zeta, t) + \frac{\omega(\zeta, t)}{\overline{\omega'(\zeta, t)}}\overline{\varphi_*'(\zeta, t)} + \overline{\psi_*(\zeta, t)} = -p(t)\,\omega(\zeta, t) \qquad \text{as} \quad |\zeta| = 1$$
$$\omega(\zeta, 0) = \omega_0(\zeta) \qquad \text{as} \quad t = 0 \tag{1.6}$$

$$\varphi_*(\zeta, t) = {}^1/_4\, c(t)[\sigma_x^\infty(t) + \sigma_y^\infty(t)]\zeta + O(\zeta^{-1}) \qquad \text{as} \quad \zeta \to \infty$$
$$\psi_*(\zeta, t) = {}^1/_2\, c(t)[\sigma_y^\infty(t) - \sigma_x^\infty(t)]\zeta + O(\zeta^{-1})$$

Here

$$\varphi_*(\zeta, t) = \varphi[\omega(\zeta, t), t], \ \psi_*(\zeta, t) = \psi[\omega(\zeta, t), t] \tag{1.7}$$

where $\omega_0(\zeta)$ is a given function.

The functions $\varphi_*(\zeta, t)$, $\psi_*(\zeta, t)$ and $\omega(\zeta, t)$ are to be determined.

It is convenient to write the kinematic compatibility condition in the form

$$2\mu \operatorname{Re}\left[\zeta \frac{\partial \overline{\omega}}{\partial t} \omega'(\zeta, t)\right] =$$

$$= \operatorname{Re}\left\{\zeta\omega'(\zeta, t)\left[\overline{\varphi_*(\zeta, t)} - \frac{\overline{\omega(\zeta, t)}}{\omega'(\zeta, t)}\varphi_*'(\zeta, t) - \psi_*(\zeta, t)\right]\right\} \qquad \text{as } |\zeta| = 1 \qquad (1.8)$$

Condition (1.8) is obtained from the following considerations (Fig. 1). Two velocities: (a) the velocity of a material point at the point O at the time t, whose complex vector $(u + iv)$ is determined by (1.1), and (b) the kinematic velocity of displacement of the point O of the contour itself L $(d\omega / dt)$ corresponding to the same value of the parameter ζ giving the position of the point O on the curve L at any time, correspond to every point O of the contour L at time t. As follows from Fig. 1 on which two infinitely close positions of the contour L are compared in the neighborhood of a point O at times t and $t + dt$, projections of the two mentioned velocity vectors on the normal n_z to the contour L at point O should be equal. Now, to prove (1.8), there just remains to find an expression for the complex vector of the unit normal n_z on L

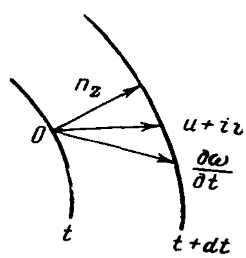

Fig. 1

$$n_z = \frac{dz}{|dz|} = \frac{\omega'(\zeta, t)}{|\omega'(\zeta, t)|}\frac{d\zeta}{|d\zeta|} = \zeta\frac{\omega'(\zeta, t)}{|\omega'(\zeta, t)|}$$

and to form the scalar product a_n of the complex vector $a = |a| \exp(i\alpha_1)$ and $n_z = \exp(i\alpha_2)$

$$a_n = |a| \cos(\alpha_1 - \alpha_2) = \operatorname{Re}(a\bar{n}_z)$$

Formulas (1.5)–(1.8) complete the formulation of the boundary value problem in the ζ plane

2. Elliptical cavity. Let us consider a class of solutions of the boundary value problem (1.5)–(1.8) in which the condition

$$2\mu \frac{\partial\omega}{\partial t} = \varphi_*(\zeta, t) - \frac{\omega(\zeta, t)}{\overline{\omega'(\zeta, t)}}\overline{\varphi_*'(\zeta, t)} - \overline{\psi_*(\zeta, t)} \qquad \text{as } |\zeta| = 1 \qquad (2.1)$$

representing the vector equality of the kinematic and material particle velocities at the boundary of the cavity, is satisfied. Condition (1.8) is hence satisfied identically.

Combining (2.1) and (1.6), we obtain

$$-p(t)\,\omega(\zeta, t) + 2\mu\frac{\partial\omega}{\partial t} = 2\varphi_*(\zeta, t) \qquad \text{as } |\zeta| = 1 \qquad (2.2)$$

By virtue of the principle of analytic continuation, (2.2) should also be satisfied in the total ζ plane

$$\varphi_*(\zeta, t) = \mu\frac{\partial\omega}{\partial t} - {}^1\!/_2\, p(t)\,\omega(\zeta, t) \qquad (2.3)$$

Substituting (2.3) into (2.1) or into (1.6) and manipulating, we find

$$\frac{\partial\omega}{\partial t}\overline{\omega'(\zeta, t)} + \omega(\zeta, t)\frac{\partial^2\overline{\omega}}{\partial t\,\partial\zeta} = -\frac{1}{\mu}\overline{\psi_*(\zeta, t)\,\omega'(\zeta, t)} \qquad \text{as } |\zeta| = 1 \qquad (2.4)$$

or

$$\mu\frac{\partial}{\partial t}[\omega(\zeta, t)\overline{\omega'(\zeta, t)}] = -\overline{\psi_*(\zeta, t)\,\omega'(\zeta, t)} \quad \text{as } |\zeta| = 1 \qquad (2.5)$$

Let us introduce an auxiliary analytic function $\Gamma(\zeta, t)$

$$\Gamma(\zeta, t) = -\frac{1}{\omega'(\zeta, t)}\left[\int_0^t \psi_*(\zeta, t)\,\omega'(\zeta, t)\,dt + \Gamma_0(\zeta)\right] \tag{2.6}$$

The function $\Gamma_0(\zeta)$ is negligible and can be considered zero. By using the function $\Gamma(\zeta, t)$ the boundary condition (2.5) can be written thus:

$$\mu\omega(\zeta, t) = \Gamma(\zeta, t) \quad \text{as} \quad |\zeta| = 1 \tag{2.7}$$

A solution of the boundary value problem (2.7) in the class of functions which have a given order $O(\zeta)$ at infinity and satisfy the symmetry conditions will be the following:

$$\omega(\zeta, t) = c(t)\,\zeta + \frac{b(t)}{\zeta}, \qquad \Gamma(\zeta, t) = \mu\left[b(t)\,\zeta + \frac{c(t)}{\zeta}\right] \tag{2.8}$$

The real functions $c(t)$ and $b(t)$ should satisfy conditions (1.6) at infinity for the functions $\varphi(\zeta, t)$ and $\psi(\zeta, t)$; by using (2.6), (2.8) and (2.3) we find the following system of first order differential equations in t for $c(t)$ and $b(t)$ from (1.6):

$$\begin{gathered} 2\mu\frac{d}{dt}[c(t)\,b(t)] + [\sigma_y^\infty(t) - \sigma_x^\infty(t)]\,[c(t)]^2 = 0 \\ 4\mu\frac{dc}{dt} - [2p(t) + \sigma_x^\infty(t) + \sigma_y^\infty(t)]\,c(t) = 0 \end{gathered} \tag{2.9}$$

where $c(t) = c_0$, $b(t) = b_0$ at $t = 0$, and b_0 and c_0 are given constants governing the shape of the cavity at the initial instant of the loading; according to (2.8) this latter is the ellipse

$$\frac{x^2}{(1+m)^2} + \frac{y^2}{(1-m)^2} = c_0^2 \quad \left(m = \frac{b_0}{c_0}\right) \tag{2.10}$$

The solution of the differential equations (2.9) which satisfies the initial conditions, is written as

$$c(t) = c_0 \exp\left[\int_0^t \lambda(\tau)\,d\tau\right]$$

$$b(t) = \left\{b_0 - c_0\int_0^t \gamma(\tau)\exp\left[2\int_0^\tau \lambda(\tau_1)\,d\tau_1\right]d\tau\right\}\exp\left[-\int_0^t \lambda(\tau)\,d\tau\right] \tag{2.11}$$

where

$$\gamma(t) = \frac{1}{2\mu}[\sigma_y^\infty(t) - \sigma_x^\infty(t)], \qquad \lambda(t) = \frac{1}{4\mu}[2p(t) + \sigma_x^\infty(t) + \sigma_y^\infty(t)]$$

By utilizing (2.8), (2.6) and (2.3), we obtain the desired functions $\varphi_*(\zeta, t)$ and $\psi_*(\zeta, t)$ in the following form:

$$\varphi_*(\zeta, t) = \frac{\sigma_x^\infty(t) + \sigma_y^\infty(t)}{4}\,c(t)\,\zeta -$$

$$-\frac{1}{4\zeta}\{2c(t)[\sigma_y^\infty(t) - \sigma_x^\infty(t)] + b(t)[4p(t) + \sigma_x^\infty(t) + \sigma_y^\infty(t)]\}$$

$$\psi_*(\zeta, t) = \frac{A(t)(\zeta^3 - \zeta^{-1}) - \zeta B(t)}{c(t)\,\zeta^2 - b(t)} \tag{2.12}$$

where

$$A(t) = \mu\gamma(t)[c(t)]^2, \qquad B(t) = 2\mu\{\lambda(t)[b^2(t) + c^2(t)] + b(t)c(t)\gamma(t)\}$$

Formulas (2.8), (2.11), (2.12) yield the exact solution of the problem of the development of a cavity from an initially elliptical one; the exact formulas taking account of material compressibility and asymmetry of the problem are easily obtained by a completely analogous method.

Let us recall that the corresponding linear problem has been solved by G. V. Kolosov and Inglis for various particular cases, and by N. I. Muskhelishvili, in general form.

As is seen, the contour of the cavity at any instant is an ellipse with center at the origin

$$\frac{x^2}{[1+m(t)]^2}+\frac{y^2}{[1-m(t)]^2}=c^2(t) \qquad \left(m=\frac{b}{c}\right) \tag{2.13}$$

Let us consider some of the most interesting particular cases of the general solution.

1°. Axisymmetric problem. At any time let the equality $\sigma_x^\infty = \sigma_y^\infty$ be satisfied. Then

$$\gamma(t)=m(t)=0, \quad \lambda(t)=\frac{1}{2\mu}[p(t)+\sigma_x^\infty(t)] \tag{2.14}$$

The cavity is the circle

$$x^2+y^2=c_0^2\exp\left\{\frac{1}{2\mu}\int_0^t[p(\tau)+\sigma_x^\infty(\tau)]\,d\tau\right\} \tag{2.15}$$

2°. Uniaxial tension of an initial circular cavity. Let $p(t)=\sigma_x^\infty(t)=0$, $b_0=0$, $\sigma_y^\infty(t)=\sigma=\text{const}$. Then

$$m(t)=-1+\exp\left(-\frac{\sigma t}{2\mu}\right), \quad c(t)=c_0\exp\left(\frac{\sigma t}{4\mu}\right) \tag{2.16}$$

The cavity contour is contracted in the direction of the x-axis and is broadened in the direction of the tension, approaching a slit along the y-axis as $t\to\infty$.

3°. Uniaxial tension of an initial slit. Let the relationships

$$b_0=c_0, \quad \sigma_x^\infty=p=0, \quad \sigma_y^\infty(t)=\sigma=\text{const}$$

be satisfied, i.e. let the body be subjected to uniaxial tension along the y-axis by a constant stress; at the initial instant there is a zero thickness slit of length $4c_0$ along the x-axis. According to the general formulas (2.11) and (2.13) we obtain

$$m=2\exp\left(-\frac{\sigma t}{2\mu}\right)-1, \qquad c=c_0\exp\left(\frac{\sigma t}{4\mu}\right) \tag{2.17}$$

Therefore, the slit is transformed into an ellipse which contracts, with time, in the x-direction and broadens in the tension direction, tending to a slit in the y-direction as $t\to\infty$. Let us present the magnitude of the stress at the most stressed point $x=(1+m)c$, $y=0$

$$\sigma_y=\sigma\,\mathrm{cth}\,\frac{\sigma t}{4\mu} \tag{2.18}$$

It is interesting that the stress at this point is independent of the applied external loading σ for small relative times (a peculiar "local plasticity")

$$\sigma_y=\frac{4\mu}{t} \qquad \text{for } \sigma t\ll 4\mu \tag{2.19}$$

The stress and velocity distribution in the neighborhood of the end of the slit at large distances compared to the radius of curvature of the ellipse at the point $x=(1+m)c$, $y=0$, but small compared with the length $4c_0$ of the initial slit, is the same for small relative loading times $\sigma t\ll 4\mu$, as the distribution of the corresponding stresses and displacements in linear elasticity theory.

The qualitative results obtained apparently are also valid to a certain extent for visco-elastic bodies.

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